

PREPARED FOR SUBMISSION TO JHEP

Integration-by-parts identities from the viewpoint of differential geometry

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ABSTRACT: We present a new method to construct integration-by-part (IBP) identities from the viewpoint of differential geometry. Vectors for generating IBP identities are reformulated as differential forms, via Poincaré duality. Using the tools of differential geometry and commutative algebra, we can efficiently find differential forms which generate on-shell IBP relation without doubled propagator. Various $D = 4$ two-loop examples are presented.

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1 Introduction

With the successful run of the Large Hadron Collider (LHC), there is an eager demand for the next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) background computation. NLO and NNLO computations involve loop-order Feynman diagrams. The number of Feynman integrals grows quickly for multi-leg and multi-loop cases. However, for each diagram, many different Feynman integrals are linearly related by the integration-by-parts (IBP) relations or symmetries, so the whole set of integrals can be reduced to a minimal set of integrals, so-called *master integrals* (MIs). This paper focuses on the geometric meaning for IBP relations and provides a new method for obtaining IBP relations.

Schematically, for a L -loop integral, the integration of a total derivative vanishes and resulting identity is called an IBP relation:

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \sum_{i=1}^L \frac{\partial}{\partial l_i^\mu} \left(\frac{v_i^\mu}{D_1^{a_1} \cdots D_k^{a_k}} \right) = 0. \quad (1.1)$$

Here v_i^μ are vectors depends on external and internal momenta.

Traditionally, various contributions to a certain amplitude are characterized by Feynman diagrams, and the final results are reduced to the form of MIs by IBP relations. In recent years, there are a lot of new methods to improve the efficiency of multi-loop diagram computation, and most of which also require the calculation of IBP identities at certain steps. Unitarity methods [1–3] relate a loop amplitude to the product of tree amplitudes, and the latter can be efficiently calculated by recursive methods [4, 5].

For example, Ossola-Papadopoulos-Pittau (OPP) method [6–11] determines the *minimal integrand basis* for one-loop Feynman diagrams algebraically via partial fraction. This method has been successfully generalized to multi-loop integrand level reduction by computational algebraic geometry [23–37]. The coefficients of the minimal integrand are therefore fixed by unitarity cuts. However, usually the integrand basis is not the minimal integral basis, so finally the results are reduced MIs by IBP relations. Multi-loop unitarity has also been systematically performed by the maximal unitarity method [14–22]. Feynman integrals are converted to contour integrals and MI coefficients can be directly extracted from residue calculations. To get the correct contour weights, in the intermediate step, IBP relations are required [14].

For multi-loop or multi-leg diagrams, in general, the computation of IBP is very heavy. For a given loop diagram, there are many IBP relations from different choices of IBP-generating vectors v_i^μ in (1.1). The desired reduction of Feynman integrals to MIs can be achieved by Gaussian elimination of IBP relations, via Laporta algorithm [41, 42]. This algorithm is used for several sophisticated programs, like AIR [43], FIRE [44] and Reduze [45]. Furthermore, Laporta algorithm can be greatly sped up by finite fields numerical sampling method [46].

A breakthrough method for generating IBP relations by Gluza, Kajda and Kosower (GKK method) [12], appeared in 2008. GKK method finds IBP relations of the integrals without doubled propagator, so only a small portion of loop integrals need to be considered. In practice, such IBP relations are found by the careful choice of IBP generating vectors v_i^μ in (1.1), via Syzygy computation [12]. Several two-loop diagrams’ IBP relations are given by this method. Furthermore, the syzygy computation can be simplified by linear algebra techniques [13]. However, GKK method does not indicate the geometric meaning of such IBP-generating vectors. It is an interesting question to ask if these vectors have any particular meaning in the loop-momentum space.

In our paper, we illustrate the geometric meaning of the IBP generating vectors for integral without doubled propagator. We reformulate such a vector as a differential form by Poincaré dual.

$$v_i^\mu \Leftrightarrow \omega, \quad (1.2)$$

where ω is a rank- $(DL - 1)$ differential form. Then we show that it is *locally* proportional to the differential form $\Omega = dD_1 \wedge \dots \wedge dD_k$,

$$\omega|_{\mathcal{S}} \propto \Omega|_{\mathcal{S}}, \quad (1.3)$$

where D_i ’s are the sets of all denominators of the Feynman integral and \mathcal{S} is the unitarity cut solution. Geometrically, ω is along the normal direction of the unitarity-cut surface.

Furthermore, we design a geometric method to generate IBP identities without doubled propagator. We consider the *primary decomposition* of the unitarity cut solutions,

$$\mathcal{S} = \bigcup_{i=1}^n \mathcal{S}_i. \quad (1.4)$$

By solving congruence equations, we construct differential form ω_i 's which is nonzero and proportional to Ω in \mathcal{S}_i , but vanishes on other branches,

$$\begin{cases} \omega_i|_{\mathcal{S}_i} = (\alpha \wedge \Omega)|_{\mathcal{S}_i} \\ \omega_i|_{\mathcal{S}_j} = 0|_{\mathcal{S}_j}, \quad j \neq i \end{cases}, \quad (1.5)$$

where α is an arbitrary non-zero $(DL - 1 - k)$ -form. We use such ω_i 's to generate the on-shell part of the IBP relations without doubled propagator. Several two-loop four-point and five-point examples are tested by our method.

This paper is organized as follows: in section 2, we reformulate IBP identities in terms of differential forms, and the condition for IBP without doubled propagator is also reformulated. In section 3, we illustrate the geometric meaning of the IBP-generating differential forms and present a new method for generating the on-shell part of IBPs. In section 4, several two-loop examples based on our algorithm are given.

2 Integration-by-Parts identities in the formalism of differential form

We consider the L -loop Feynman integral,

$$I_{\{a_1, \dots, a_k\}}[N] = \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \frac{N}{D_1^{a_1} \cdots D_k^{a_k}}. \quad (2.1)$$

where N is a polynomial in loop momenta. The integrand reduction and unitarity solution structure has been studied by algebraic geometry methods [25, 26]. In the following discussion, we will frequently use these algebraic geometry methods. The mathematical notations are summarized in the Appendix and the algebraic geometry reference is [49].

We find that it is convenient to rewrite IBP relations (1.1) in terms of differential forms. By Poincaré dual, the $(D \cdot L)$ -dimensional vector v_i^μ is dual to a $D \cdot L - 1$ differential form ω . Explicitly,

$$\omega_{i_1 \dots i_{(DL-1)}} \equiv \epsilon_{i_1 \dots i_{(DL-1)} i_{DL}} v^{i_{DL}}, \quad (2.2)$$

where $\epsilon_{i_1 \dots i_{(DL-1)} i_{DL}}$ is the Levi-Civita symbol. In most of the following discussion, we use the notations of differential forms, since it is convenient to write down the exterior derivative and wedge products. We call a differential form *polynomial-valued*, if all the components are polynomials in loop momenta, in the momentum-coordinate basis. Note that this definition is consistent with linear transformation of loop momenta.

The total derivative in (1.1) can be dually written as,

$$\frac{\partial}{\partial l_i^\mu} \left(\frac{v_i^\mu}{D_1^{a_1} \cdots D_k^{a_k}} \right) \Leftrightarrow d \left(\frac{\omega}{D_1^{a_1} \cdots D_k^{a_k}} \right). \quad (2.3)$$

So the IBP relation is

$$\int \frac{d\omega}{D_1^{a_1} \dots D_k^{a_k}} - \sum_{i=1}^k a_i \int \frac{dD_i \wedge \omega}{D_1^{a_1} \dots D_i^{a_i+1} \dots D_k^{a_k}} = 0. \quad (2.4)$$

Different choices of v_i^μ , or ω lead to different IBPs. One particularly interesting class of IBPs is *IBPs without doubled propagator*, which is described in the next subsection.

2.1 IBPs without doubled propagator

For a Feynman integral from Feynman rules, the powers of the denominators D_1, \dots, D_k in (2.1) are usually one or zero, i.e., $a_i = 0, 1, i = 1, \dots, k$. We call such an integral, *integral without doubled propagator*. We are interested in *IBPs without doubled propagators*, which is an IBP whose terms are integrals without doubled propagator.

We make an ansatz for an IBP without doubled propagator,

$$\int d\left(\frac{\omega}{D_1 \dots D_k}\right) = 0, \quad (2.5)$$

where ω is a polynomial-valued $(DL - 1)$ -form. Usually, the expansion of (1.1) contains integrals with double propagators, because,

$$d\left(\frac{1}{D_i}\right) = -\frac{dD_i}{D_i^2}. \quad (2.6)$$

However, a particular choice of ω can remove the double power if,

$$dD_i \wedge \omega = f_i D_i dl_1^0 \wedge \dots \wedge dl_L^{D-1}, \quad i = 1, \dots, j \quad (2.7)$$

where f_i is a polynomial.

2.2 On-shell part of IBPs

Sometimes we only focus on Feynman diagrams without pinched legs, i.e., $a_i \geq 1, i = 1, \dots, k$. We call the corresponding integrals *leading integrals*. On the other hand, we call integrals with at least one $a_i < 1$ *simpler integrals*. If we only keep the leading integrals in an IBP relation, then the resulting formula

$$\sum_i c_i I_{a_{i,1}, \dots, a_{i,k}}[N_i] + \dots = 0, \quad (2.8)$$

is called an *on-shell IBP relation*. $a_{i,j} > 0, \forall i, j$. Here “...” denotes the *simpler integrals*, and N_i ’s are polynomial numerators.

In this paper, we consider the on-shell IBP without double propagators, namely,

$$\sum_i c_i I_{1, \dots, 1}[N_i] + \dots = 0, \quad (2.9)$$

For the ansatz (2.5) to generate an on-shell IBP without doubled propagator, it is sufficient that,

$$dD_i \wedge \omega = \sum_j f_{ij} D_j dl_1^0 \wedge \dots \wedge dl_L^{D-1}, \quad i = 1, \dots, j \quad (2.10)$$

where each f_{ij} is a polynomial. ω generates the IBP,

$$0 = \int d\left(\frac{\omega}{D_1 \dots D_k}\right) = \int \frac{d\omega}{D_1 \dots D_k} - \sum_{i=1}^k \sum_{j=1}^k \int \frac{f_{ij} D_j dl_1^0 \wedge \dots \wedge dl_L^{D-1}}{D_1 \dots D_i^2 \dots D_k}, \quad (2.11)$$

Pick up the on-shell part, we have

$$0 = \int \frac{d\omega}{D_1 \dots D_k} - \sum_{i=1}^k \int \frac{f_{ii} dl_1^0 \wedge \dots \wedge dl_L^{D-1}}{D_1 \dots D_k} + \dots, \quad (2.12)$$

where \dots stands for simpler integrals. Note that this condition (2.10) is weaker than the condition (2.7).

Furthermore, from (2.12), we have the following lemma,

Lemma 1. *If all components of ω are in the ideal $I = \langle D_1, \dots, D_k \rangle$, then it generates an IBP identity whose on-shell part is trivial.*

Proof. Let $\omega' = \sum_{i=1}^m w_i dx_1 \wedge \dots \wedge \hat{dx}_i \wedge \dots \wedge dx_m$, where $m = LD$ and $\{x_1, \dots, x_m\}$ denote the loop momenta $\{l_1^0, \dots, l_L^{D-1}\}$. Suppose that every w_i is in I , i.e., $w_i = \sum_{j=1}^k g_{ij} D_j$. Hence,

$$\begin{aligned} 0 &= \int d\left(\frac{\omega}{D_1 \dots D_k}\right) = \sum_{i=1}^m \sum_{j=1}^k \int d\left(\frac{g_{ij} D_j dx_1 \wedge \dots \wedge \hat{dx}_i \wedge \dots \wedge dx_m}{D_1 \dots D_k}\right) \\ &= \sum_{i=1}^m \sum_{j=1}^k \int d\left(\frac{g_{ij} dx_1 \wedge \dots \wedge \hat{dx}_i \wedge \dots \wedge dx_m}{D_1 \dots \hat{D}_j \dots D_k}\right). \end{aligned} \quad (2.13)$$

From the expansion of the expression, it is clear that each term misses one of the denominators. Therefore, ω' generates the IBP,

$$0 = 0 + \dots, \quad (2.14)$$

where \dots stands for simpler integrals. The on-shell part is trivial. \square

From this lemma, if two rank- $DL - 1$ forms ω_1 and ω differ by such an ω' , then ω_1 and ω_2 generate the same on-shell IBP. If an ω satisfying (2.10), then $f\omega$ also satisfies (2.10). Here f is a polynomial in loop momenta. So we can obtain more IBPs without doubled propagator, by multiplying various f 's. Note that by Lemma 1, only when f is a polynomial in *irreducible scalar products*, the resulting $f\omega$ generates a non-trivial on-shell IBP.

3 A method to construct on-shell IBPs without doubled propagator

We reformulate (2.10) from the viewpoint of algebraic geometry, and then illustrate how to find the solution to (2.10) with computational algebraic geometry method.

3.1 A condition for on-shell IBPs without doubled propagator

With the background of algebraic geometry, we can reformulate the condition (2.10) as the differential geometry constraint in Proposition 2.

Proposition 1. *For an ω in (2.5) to generate an on shell IBP without doubled propagator, it is necessary that for each point on the cut solution, at the corresponding cotangent space,*

$$(dD_i \wedge \omega)|_P = 0, \quad \forall P \in \mathcal{Z}(I). \quad (3.1)$$

If the ideal generate by the denominators is radical, then this condition is also sufficient.

Proof. By the definition, all D_i vanish on $\mathcal{S} = \mathcal{Z}(I)$. So $\forall P \in \mathcal{Z}(I)$, $(dD_i \wedge \omega)|_P = 0$. On the other hand,

$$(dD_i \wedge \omega) = F_i dl_1^0 \wedge \dots \wedge dl_L^{D-1}, \quad i = 1, \dots, k \quad (3.2)$$

where each F_i is a polynomial. (3.1) means that F_i vanish everywhere on \mathcal{S} . So by Hilbert's Nullstellensatz, $F_i \in \sqrt{I}$. If I is radical, then $F_i \in I$ and so $F_i = \sum_j f_{ij} D_j$. \square

To get some insights of (3.1), we consider the cotangent space at P . We consider general case, for which the cut equation system is non-degenerate, i.e.,

$$\dim \mathcal{S}_i = DL - k, \quad i = 1, \dots, n \quad (3.3)$$

where k is the number of denominators. If P is a *non-singular point*, i.e., the Jacobian

$$J = \det \left(\frac{\partial D_i}{\partial x_j} \right) |_P. \quad (3.4)$$

has the rank k , then it is clearly that

$$(dD_1 \wedge \dots \wedge dD_k)|_P \neq 0. \quad (3.5)$$

Therefore we have the following proposition,

Proposition 2. *If $k \leq DL - 1$ and all cut solutions have the dimension $DL - k$, for an ω in (2.5) to generate an on shell IBP without doubled propagator, it is necessary that for each non-singular point P on the cut solution, at the cotangent space,*

$$\omega|_P = (\alpha \wedge D_1 \wedge \dots \wedge D_k)|_P. \quad (3.6)$$

where α is a $(DL - k - 1)$ form.

Proof. Since at the non-singular point P , the Jacobian is non-zero. So locally we can choose a coordinator system, (y_1, \dots, y_{DL}) such that,

$$y_1 = D_1, \quad \dots, \quad y_k = D_k. \quad (3.7)$$

Expand $\omega|_P$ in this coordinator system. If $\omega|_P$ contains a component proportional to $dy_1 \wedge \dots \wedge \hat{dy}_i \wedge \dots \wedge dy_n$ and $i \leq k$, then

$$(dD_i \wedge \omega)|_P \neq 0. \quad (3.8)$$

This is a violation to Proposition 1. Collecting all terms proportional to $dy_1 \wedge \dots \wedge \hat{dy}_i \wedge \dots \wedge dy_n$ and $i > k$, this lemma is clear. \square

Generically, the singular points on \mathcal{S} only form a subset with lower dimension. So for “almost all points” on \mathcal{S} , ω is proportional to $dD_1 \wedge \dots \wedge dD_k$. We may have an explicit ansatz,

$$\omega = \alpha \wedge dD_1 \wedge \dots \wedge dD_k. \quad (3.9)$$

Here α is a polynomial-valued differential form. This indeed generates an on-shell IBP relation without double propagator. However, this form may not generate enough IBP relations, since proposition 1 is only a local condition while (3.9) has a global expression.

We may generalize (3.9) as: a polynomial-valued differential form ω which *locally* has the form,

$$\omega|_{\mathcal{S}_i} = \alpha_i \wedge dD_1 \wedge \dots \wedge dD_k. \quad (3.10)$$

on each branch \mathcal{S}_i . α_i 's are different polynomial $(DL - k - 1)$ -forms on different branches. Then there are two questions,

- Given a set of α_i 's, does such a polynomial-valued ω exist?
- Given a set of α_i 's, is there an algorithm to find such an ω ?

These questions will be answered in the next section, explicitly in Theorem 1, by solving *congruence equations*.

3.2 Local form and congruence equations

To study the behaviour of a differential form near the cut, we use the tool of Gröbner basis and polynomial divisions. Recall that I has the primary decomposition $I = I_1 \cap \dots \cap I_n$. Let $G(I)$ be the Gröbner basis of I , and $G(I_i)$ be the Gröbner basis of I_i . We denote the equivalent classes $[\]$ and $[\]_i$ as,

$$[f] = [g], \quad \text{if } f - g \in I, \quad (3.11)$$

$$[f]_i = [g]_i, \quad \text{if } f - g \in I_i. \quad (3.12)$$

Intuitively, these equivalent classes characterise the limit of the polynomials approaching the cut manifold. In practise, the unique representative for $[f]$ (or $[f]_i$) can be chose as the remainder of the polynomial division of f over $G(I)$ (or $G(I_i)$).

Here we generalize the equivalent classes to polynomial-valued differential forms. Two differential forms α and β are in the same equivalent classes, if and only if α and β are of the same rank and all polynomial components are in the same equivalent classes. We still use $[\]$ and $[\]_i$ for differential forms.

Then we rewrite the condition (3.10) as,

$$[\omega]_i = [\alpha_i \wedge dD_1 \wedge \dots \wedge dD_k]_i. \quad (3.13)$$

For a large classes of diagrams, given an arbitrary set of α_i 's, such differential form ω exists. We have the following theorem,

Theorem 1. Let $I = \langle D_1, \dots, D_k \rangle$ be an ideal in the ring $\mathbb{C}[x_1, \dots, x_m]$. $I = I_1 \cap \dots \cap I_n$ is its primary decomposition and $J_i = \cap_{j=1}^i I_j$. Suppose that (1) for each component $\dim \mathcal{Z}(I_i) = m - k$ (2) Each $(J_i + I_{i+1})$ is a radical ideal, $i = 1, \dots, n - 1$. Then given an arbitrary set of rank- $(m - k - 1)$ polynomial-valued forms, α_i , there exists a rank- $(m - 1)$ form ω such that,

$$[\omega]_i = [\alpha_i \wedge dD_1 \wedge \dots \wedge dD_k]_i. \quad (3.14)$$

Proof. We construct ω explicitly by solving congruence equations. Define $v_i = \alpha_i \wedge dD_1 \wedge \dots \wedge dD_k$. First, the ideal $I_1 + I_2$'s zero locus is $\mathcal{Z}(I_1 + I_2) = \mathcal{Z}(I_1) \cap \mathcal{Z}(I_2)$, which are all singular points on the algebraic set $\mathcal{Z}(I)$. Since $\dim \mathcal{Z}(I_i) = m - k$, the Jacobian $\partial D_i / \partial x_j$'s rank is strictly less than k on $\mathcal{Z}(I_1 + I_2)$. In other words, $dD_1 \wedge \dots \wedge dD_k$ vanishes on $\mathcal{Z}(I_1 + I_2)$. Hence $v_1 - v_2$ vanishes on $\mathcal{Z}(I_1 + I_2)$. Then by using Hilbert Nullstellensatz for each component and the condition that $I_1 + I_2$ is radical, $v_1 - v_2$ is in $I_1 + I_2$, i.e.,

$$v_1 - v_2 = a_1 + a_2, \quad a_1 \in I_1, \quad a_2 \in I_2 \quad (3.15)$$

Define $v_{12} = v_1 - a_1$. Then $[v_{12}]_1 = [v_1]_1$ and $[v_{12}]_2 = [v_2]_2$. Then by induction, we have a differential form $v_{1\dots i}$ such that $[v_{1\dots i}]_j = [v_j]_j$, $\forall 1 \leq j \leq i$. The zero locus of $J_i + I_{i+1}$ is,

$$\mathcal{Z}(J_i + I_{i+1}) = \bigcup_{j=1}^i (\mathcal{Z}(I_j) \cap \mathcal{Z}(I_{i+1})). \quad (3.16)$$

which are also singular points on the algebraic set $\mathcal{Z}(I)$. Since $[v_{1\dots i}]_j = [\alpha_j \wedge dD_1 \wedge \dots \wedge dD_k]_j$, $v_{1\dots i}$ vanishes on $\mathcal{Z}(I_j) \cap \mathcal{Z}(I_{i+1})$. Hence both $v_{1\dots i}$ and v_{i+1} vanish on $\mathcal{Z}(J_i + I_{i+1})$. Then by using Hilbert Nullstellensatz, we obtain the differential form $v_{1\dots(i+1)}$. Finally we denote $v_{1\dots n} = \omega$. \square

A large classes of 4D high-loop diagrams satisfy two conditions in the above proposition. So we can construct ω for the IBP without doubled propagator. The proof itself provides the algorithm for obtaining ω . This algorithm is realized by our Mathematica and Macaulay2 [50] package, MATHEMATICAM2.¹

Remark 1. Note that in practice, after obtaining the differential form ω which satisfies (3.1), there may exist further simplification. The form ω may factorize as,

$$\omega = f\omega'. \quad (3.17)$$

where f is a polynomial in loop momenta and ω' is a polynomial-valued form. If ω satisfies (3.1), there is no guarantee that ω' also satisfies (3.1). However, if accidentally ω' satisfies (3.1), we can instead use ω' to generate an IBP without doubled propagator.

4 Examples

In this section, we demonstrate our method by several 4D two-loop examples. In each case, we generate the 4D on-shell part of the IBP identities by our differential geometry method, via local form and congruence equations. To simplify the process, we combine integrand reduction method and our differential geometry approach for IBP computations.

¹This package can be downloaded from <http://www.nbi.dk/~zhang/MathematicaM2.html>.

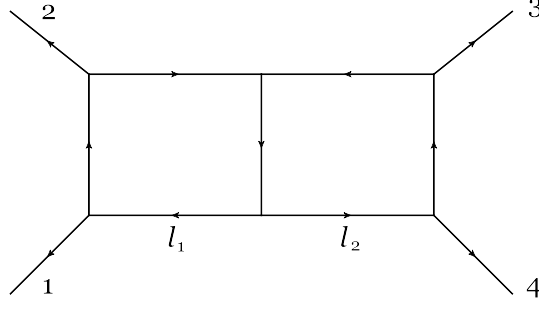


Figure 1. Planar double box with 4 massless legs

4.1 Planar double box

Consider the $4D$ planar double box with 4 massless legs, p_1, p_2, p_3 and p_4 . The two loop momenta are l_1 and l_2 . There are 7 denominators for double box integrals,

$$\begin{aligned} D_1 &= l_1^2, & D_2 &= (l_1 - p_1)^2, & D_3 &= (l_1 - p_1 - p_2)^2, \\ D_4 &= (l_2 - p_3 - p_4)^2, & D_5 &= (l_2 - p_4)^2, & D_6 &= l_2^2, & D_7 &= (l_1 + l_2)^2. \end{aligned} \quad (4.1)$$

Instead of using Minkowski components of l_1 and l_2 , we use van Neerven-Vermaseren basis,

$$\begin{aligned} x_1 &= l_1 \cdot p_1, & x_2 &= l_1 \cdot p_2, & x_3 &= l_1 \cdot p_4, & x_4 &= l_1 \cdot \omega, \\ y_1 &= l_2 \cdot p_1, & y_2 &= l_2 \cdot p_2, & y_3 &= l_2 \cdot p_4, & y_4 &= l_2 \cdot \omega. \end{aligned} \quad (4.2)$$

where ω is the vector which is perpendicular to all external legs and $\omega^2 = tu/s$. The denominators have the parity symmetry,

$$x_4 \leftrightarrow -x_4, \quad y_4 \leftrightarrow -y_4. \quad (4.3)$$

Define the ideal $I \equiv \langle D_1, \dots, D_7 \rangle$. The ISPs are $\{x_3, x_4, y_1, y_4\}$. Integrals with numerators linear in x_4 or y_4 are spurious, i.e., vanish by the orthogonal property of ω .

The $4D$ double box cut has 6 branches,

$$I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6, \quad (4.4)$$

where,

$$I_1 = \langle x_1, -s - 2y_1 - 2y_2, s - 2x_2, y_3, x_3, t - 2y_1 + 2y_4, 2x_4 - t \rangle, \quad (4.5)$$

$$I_2 = \langle y_1, x_1, s + 2y_2, s - 2x_2, y_3, t + 2y_4, -t + 2x_3 + 2x_4 \rangle, \quad (4.6)$$

$$I_3 = \langle x_1, -s - 2y_1 - 2y_2, s - 2x_2, y_3, x_3, -t + 2y_1 + 2y_4, t + 2x_4 \rangle, \quad (4.7)$$

$$I_4 = \langle y_1, x_1, s + 2y_2, s - 2x_2, y_3, 2y_4 - t, t - 2x_3 + 2x_4 \rangle, \quad (4.8)$$

$$\begin{aligned} I_5 &= \langle x_1, s + 2y_1 + 2y_2, s - 2x_2, y_3, -st + 2sx_3 + 2sy_1 + 4x_3y_1, \\ &\quad t - 2y_1 + 2y_4, t - 2x_3 + 2x_4 \rangle, \end{aligned} \quad (4.9)$$

$$\begin{aligned} I_6 &= \langle x_1, s + 2y_1 + 2y_2, s - 2x_2, y_3, -st + 2sx_3 + 2sy_1 + 4x_3y_1, \\ &\quad -t + 2y_1 + 2y_4, -t + 2x_3 + 2x_4 \rangle \end{aligned} \quad (4.10)$$

Note that under the parity symmetry (4.3), the primary ideals are permuted,

$$I_1 \leftrightarrow I_3, \quad I_2 \leftrightarrow I_4, \quad I_5 \leftrightarrow I_6 \quad (4.11)$$

We can first carry out the integrand reduction for double-box numerators. The irreducible numerator terms have the form,

$$x_3^m y_1^n x_4^a y_4^b. \quad (4.12)$$

The renormalizability condition requires that $0 \leq m + a \leq 4, 0 \leq n + b \leq 4, 0 \leq m + n + a + b \leq 6$. Furthermore, the Gröbner basis and polynomial division method ² [25] determines that, the integrand basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, contains 32 terms,

$$\mathcal{B}_1 = \{x_3^4 y_1, x_3 y_1^4, x_3^4, x_3^3 y_1, x_3 y_1^3, y_1^4, x_3^3, x_3^2 y_1, x_3 y_1^2, y_1^3, x_3^2, x_3 y_1, y_1^2, x_3, y_1, 1\} \quad (4.13)$$

and

$$\begin{aligned} \mathcal{B}_2 = \{ & x_4, x_3 x_4, x_3^2 x_4, x_3^3 x_4, x_4 y_1, y_4, x_3 y_4, x_3^2 y_4, x_3^3 y_4, x_3^4 y_4, y_1 y_4, x_3 y_1 y_4, y_1^2 y_4, \\ & x_3 y_1^2 y_4, y_1^3 y_4, x_3 y_1^3 y_4 \}. \end{aligned} \quad (4.14)$$

Note all terms in \mathcal{B}_2 are spurious. So we focus on further reducing the 16 terms in \mathcal{B}_1 via IBPs. We divide our algorithm in several steps,

1. Evaluate $\Omega = dD_1 \wedge \dots \wedge dD_7$ and the local forms $[\Omega]_i$. Direct computation gives,

$$\begin{aligned} \Omega = \frac{128s}{t^3(s+t)^3} & \left((s(x_4(y_1+y_3) - y_4(x_1+x_3)) + t(y_4(x_2-x_1) + x_4(y_1-y_2))) \right. \\ & (s(y_1+y_3) + t(y_1+y_2+2y_3)) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 \\ & + sy_4(s(y_4(x_1+x_3) - x_4(y_1+y_3)) + t(y_4(x_1-x_2) + x_4(y_2-y_1))) \\ & dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 \\ & + sy_4(s(y_4(x_1+x_3) - x_4(y_1+y_3)) + t(y_4(x_1-x_2) + x_4(y_2-y_1))) \\ & dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4 \\ & - (s(y_4(x_1+x_3) - x_4(y_1+y_3)) + t(y_4(x_1+x_2+2x_3) - x_4(y_1+y_2+2y_3))) \\ & (s(x_1+x_3) + t(x_1-x_2)) dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 \\ & \left. - sx_4(s(x_4(y_1+y_3) - y_4(x_1+x_3)) + t(x_4(y_1+y_2+2y_3) - y_4(x_1+x_2+2x_3))) \right. \\ & \left. dx_1 \wedge dx_2 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 \right). \end{aligned} \quad (4.15)$$

The canonical representative of $[\Omega]_i$ is obtained by polynomial division. For example, on the first branch,

$$\begin{aligned} [\Omega]_1 = & -\frac{64s^2 y_1(t-2y_1)}{t^2(s+t)^2} (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 \\ & - dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 - dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4). \end{aligned} \quad (4.16)$$

²The package for integrand reduction can be downloaded from <http://www.nbi.dk/~zhang/BasisDet.html>.

2. Verify that the two conditions in Theorem 1 hold. In this case, $k = 7$ and $m = DL = 8$, so $m - k = 1$. On the other hand, all six branches are one-dimensional. Furthermore, define $J_i = \cap_{j=1}^i I_i$. Directly commutative algebra computations indicate that $J_i + I_{i+1}$ is radical, for $i = 1, 2, 3, 4, 5$.
3. Solve the congruence equations in the polynomial ring. Let $\eta_i, i = 1, \dots, 6$ be 7-forms satisfy the following equations,

$$\begin{cases} [\eta_i]_j = [\Omega]_j & j = i \\ [\eta_i]_j = 0 & j \neq i, \quad j = 1, \dots, 6 \end{cases} \quad (4.17)$$

The solution for η_i 's can be quickly obtained by our package MATHEMATICAM2. For example,

$$\begin{aligned} \eta_1 = & -\frac{16s(s(t(x_4 + 2y_1 + y_4) - 2(x_3(2y_1 + y_4) + y_1(x_4 + 2(y_1 + y_4)))) - 8x_3y_1(y_1 + y_4))}{t^2(s + t)^2} \\ & (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 - dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 \\ & - dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4). \end{aligned} \quad (4.18)$$

It is easy to check that,

$$[\eta_1]_1 = [\Omega]_1, \quad [\eta_1]_2 = [\eta_1]_3 = [\eta_1]_4 = [\eta_1]_5 = [\eta_1]_6 = 0. \quad (4.19)$$

4. Find all the IBP relations generated by $f\eta_j$ according to (2.12), where $f \in B$ is a term from the integrand basis. For $4D$ double box case, the process can be sped up by using the parity symmetry. Define the 7-forms according to the permutation of primary ideals,

$$v_1 = \eta_1 + \eta_3, \quad v_2 = \eta_2 + \eta_4, \quad v_3 = \eta_5 + \eta_6 \quad (4.20)$$

Then v_i 's, $i = 1, 2, 3$ are even under the parity symmetry. Hence, we can consider IBP relations generated by $f v_j$, where $f \in B_1$. In this way, we avoid the redundancy from spurious terms. For example, explicitly,

$$\begin{aligned} v_1 = & \frac{32s}{t^2(s + t)^2} \left(- (s(t(x_4 + y_4) - 2(x_3y_4 + x_4y_1 + 2y_1y_4)) - 8x_3y_1y_4) \right. \\ & dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 - 2y_1(s(2(x_3 + y_1) - t) + 4x_3y_1) \\ & dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4 - 2y_1(s(2(x_3 + y_1) - t) + 4x_3y_1) \\ & \left. dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 \right). \end{aligned} \quad (4.21)$$

Consider the form $w = y_1 v_1$.

$$dw = -\frac{32sy_1(s(-5t + 10x_3 + 16y_1) + 32x_3y_1)}{t^2(s + t)^2} \mathbf{m} \quad (4.22)$$

Here \mathbf{m} is the measure, $\mathbf{m} = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4$. Furthermore, it is clear that $dD_i \wedge \omega = f_{ij} D_j \mathbf{m}$. The related components are,

$$f_{11} = 0, \quad f_{22} = 0, \quad f_{33} = 0, \quad (4.23)$$

$$f_{44} = \frac{16sy_1(st^2 - 2stx_3 - 6sty_1 - 4sx_3y_1 + 8sy_1^2 - 16tx_3y_1 + 16x_3y_1^2)}{t^2(s+t)^3} \quad (4.24)$$

$$f_{55} = \frac{16sy_1}{t^3(s+t)^3} (s^2t^2 - 2s^2tx_3 - 6s^2ty_1 - 4s^2x_3y_1 - 8s^2y_1^2 - 16stx_3y_1 - 16sty_1^2 - 16sx_3y_1^2 - 32tx_3y_1^2) \quad (4.25)$$

$$f_{66} = \frac{16sy_1(st^2 - 6stx_3 - 6sty_1 + 4sx_3y_1 + 8sy_1^2 - 16tx_3y_1 + 16x_3y_1^2)}{t^3(s+t)^2} \quad (4.26)$$

$$f_{77} = \frac{64sy_1(st - sx_3 - 3sy_1 - 4x_3y_1)}{t^2(s+t)^2} \quad (4.27)$$

Using (2.12), we get one IBP relation,

$$\begin{aligned} & -4I_{\text{dbox}}[(l_1 \cdot p_4)(l_2 \cdot p_1)^2] - 2sI_{\text{dbox}}[(l_1 \cdot p_4)(l_2 \cdot p_1)] \\ & - 2sI_{\text{dbox}}[(l_2 \cdot p_1)^2] + stI_{\text{dbox}}[(l_2 \cdot p_1)] + \dots = 0 \end{aligned} \quad (4.28)$$

Using this algorithm, we find that both v_1 and v_2 provide 3 IBP relations, while v_3 provides 6 IBP relations. These relations are linearly independent. So our method reduces the number of double box integrals from 16 to $16 - 12 = 4$. The resulting 4 integrals can be chosen as

$$I_{\text{dbox}}[1], \quad I_{\text{dbox}}[l_1 \cdot p_4], \quad I_{\text{dbox}}[l_2 \cdot p_1], \quad I_{\text{dbox}}[(l_1 \cdot p_4)(l_2 \cdot p_1)] \quad (4.29)$$

Furthermore, the symmetry of double box determines that,

$$I_{\text{dbox}}[l_1 \cdot p_4] = I_{\text{dbox}}[l_2 \cdot p_1]. \quad (4.30)$$

So we reduce the number of independent integrals to 3. Our 4D formalism misses one IBP relation which can be obtained from the D -dimensional formalism,

$$I_{\text{dbox}}[(l_1 \cdot p_4)(l_2 \cdot p_1)] = \frac{1}{8}stI_{\text{dbox}}[1] - \frac{3}{4}sI_{\text{dbox}}[l_1 \cdot p_4] + \dots \quad (4.31)$$

This identity occurs in the $O(\epsilon)$ -order in a D -dimensional IBP relation. So it cannot be detected by the pure 4D IBP formalism. Including this missing IBP, all integrals for 4D double box are reduced to two master integrals,

$$I_{\text{dbox}}[1], \quad I_{\text{dbox}}[l_1 \cdot p_4], \quad (4.32)$$

and we verified that the result is consistent with the 4D limit of the output of FIRE. For

example,

$$I_{\text{dbox}}[(l_1 \cdot p_4)^2] = \frac{t}{2} I_{\text{dbox}}[l_1 \cdot p_4] + \dots, \quad (4.33)$$

$$I_{\text{dbox}}[(l_1 \cdot p_4)^3] = \frac{t^2}{4} I_{\text{dbox}}[l_1 \cdot p_4] + \dots, \quad (4.34)$$

$$I_{\text{dbox}}[(l_1 \cdot p_4)^4] = \frac{t^3}{8} I_{\text{dbox}}[l_1 \cdot p_4] + \dots, \quad (4.35)$$

$$I_{\text{dbox}}[(l_1 \cdot p_4)^2(l_2 \cdot p_1)] = -\frac{s^2 t}{16} I_{\text{dbox}}[1] + \frac{3s^2}{8} I_{\text{dbox}}[l_1 \cdot p_4] + \dots, \quad (4.36)$$

$$I_{\text{dbox}}[(l_1 \cdot p_4)^3(l_2 \cdot p_1)] = \frac{s^3 t}{32} I_{\text{dbox}}[1] - \frac{3s^3}{16} I_{\text{dbox}}[l_1 \cdot p_4] + \dots \quad (4.37)$$

4.1.1 Comparison with GKK method

It is interesting to see the relation between our method and GKK method [12]. GKK method solves syzygy equations for generating vectors without doubled propagator. We treat the generating vector v as a dual differential form ω . On each branch it is easy to find the local form of ω and finally we combine local forms together by solving congruence equations. So far, our method is limited to $4D$ and the on-shell part.

We compare the $4D$ and the on-shell part of the generating vectors for double box from GKK method. There are three such vectors in [12] for double box with four massless legs, namely

$$v_{\text{GKK}}^{(1)}, \quad v_{\text{GKK}}^{(2)}, \quad v_{\text{GKK}}^{(3)} \quad (4.38)$$

To compare these with our result, we take the Poincaré dual of these vectors, namely $\omega_{\text{GKK}}^{(1)}$, $\omega_{\text{GKK}}^{(2)}$ and $\omega_{\text{GKK}}^{(3)}$. Then we can verify that the on-shell part is related to our result as,

$$[\omega_{\text{GKK}}^{(1)}] = \frac{t^2(s+t)^2}{64s^2} ([\eta_1] + [\eta_2] + [\eta_3] + [\eta_4] - [\eta_5] - [\eta_6]), \quad (4.39)$$

$$[\omega_{\text{GKK}}^{(2)}] = \frac{t^2(s+t)^2}{64s} (-[\eta_1] + [\eta_2] - [\eta_3] + [\eta_4] - [\eta_5] - [\eta_6]), \quad (4.40)$$

$$[\omega_{\text{GKK}}^{(3)}] = \frac{t^2(s+t)^2}{64s} \left(\frac{s+2(l_2 \cdot k_1)}{s} [\eta_1] - [\eta_2] + \frac{s+2(l_2 \cdot k_1)}{s} [\eta_3] \right. \quad (4.41)$$

$$\left. -[\eta_4] - \frac{s+2(l_2 \cdot k_1)}{s} [\eta_5] - \frac{s+2(l_2 \cdot k_1)}{s} [\eta_6] \right). \quad (4.42)$$

So on-shell, $\omega_{\text{GKK}}^{(i)}$'s are the linear combination of the differential form η_i 's. (The overall factor $t^2(s+t)^2/(64s)$ comes from the normalization and has no significant meaning.) The coefficients are the same for branch pairs (under the parity symmetry), so the spurious terms drop out in the IBP calculation.

Therefore, our method reproduces the $4D$ on-shell part of the double box result from GKK.

4.2 Non-planar crossed box

Our method also works for non-planar diagrams. For example, consider the $4D$ crossed box with 4 massless legs, p_1, p_2, p_3 and p_4 . The two loop momenta are l_1 and l_2 .

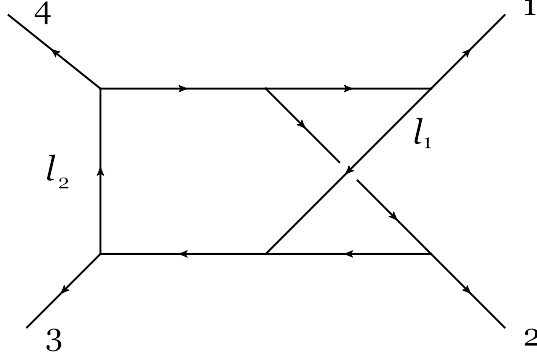


Figure 2. Non-planar double box with 4 massless legs

There are 7 denominators for crossed box integrals,

$$\begin{aligned} D_1 &= (l_1 + p_1)^2, & D_2 &= l_1^2, & D_3 &= (l_2 + p_3)^2, \\ D_4 &= l_2^2, & D_5 &= (l_2 - p_4)^2, & D_6 &= (l_2 - l_1 + p_2 + p_3)^2, & D_7 &= (l_2 - l_1 + p_3)^2. \end{aligned} \quad (4.43)$$

Again we use van Neerven-Vermaseren basis,

$$\begin{aligned} x_1 &= l_1 \cdot p_1, & x_2 &= l_1 \cdot p_2, & x_3 &= l_1 \cdot p_3, & x_4 &= l_1 \cdot \omega, \\ y_1 &= l_2 \cdot p_1, & y_2 &= l_2 \cdot p_2, & y_3 &= l_2 \cdot p_3, & y_4 &= l_2 \cdot \omega. \end{aligned} \quad (4.44)$$

where ω is the vector which is perpendicular to all external legs and $\omega^2 = tu/s$. Again, the denominators have the parity symmetry,

$$x_4 \leftrightarrow -x_4, \quad y_4 \leftrightarrow -y_4. \quad (4.45)$$

Define the ideal $I \equiv \langle D_1, \dots, D_7 \rangle$. The ISPs are $\{x_3, x_4, y_1, y_4\}$. Integrals with numerators linear in x_4 or y_4 are spurious.

This diagram has the following symmetry,

$$l_1 \rightarrow l_1 - l_2 + p_1 + p_4, \quad l_2 \rightarrow -l_2, \quad (4.46)$$

$$p_1 \rightarrow p_2, \quad p_2 \rightarrow p_1, \quad p_3 \rightarrow p_4, \quad p_4 \rightarrow p_3. \quad (4.47)$$

The $4D$ crossed box cut has 8 branches,

$$I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6 \cap I_7 \cap I_8, \quad (4.48)$$

where,

$$I_1 = \langle -t + 2x_2 - 2y_2, y_1 + y_2, x_1, y_3, x_3 + y_2, y_2 + y_4, -\frac{t^2}{s} - \frac{2ty_2}{s} - t + 2x_4 \rangle, \quad (4.49)$$

$$I_2 = \langle -t + 2x_2 - 2y_2, y_1 + y_2, x_1, y_3, x_3 + y_2, y_4 - y_2, \frac{t^2}{s} + \frac{2ty_2}{s} + t + 2x_4 \rangle, \quad (4.50)$$

$$I_3 = \langle t + 2y_2, x_2, 2y_1 - t, x_1, y_3, 2y_4 - t, x_4 - x_3 \rangle, \quad (4.51)$$

$$I_4 = \langle t + 2y_2, x_2, 2y_1 - t, x_1, y_3, t + 2y_4, x_3 + x_4 \rangle, \quad (4.52)$$

$$I_5 = \langle -t + 2x_2 - 2y_2, y_1 + y_2, x_1, y_3, x_3, y_2 + y_4, \frac{t^2}{s} + y_2(\frac{2t}{s} + 2) + t + 2x_4 \rangle, \quad (4.53)$$

$$I_6 = \langle -t + 2x_2 - 2y_2, y_1 + y_2, x_1, y_3, x_3, y_4 - y_2, -\frac{t^2}{s} + y_2(-\frac{2t}{s} - 2) - t + 2x_4 \rangle, \quad (4.54)$$

$$I_7 = \langle s + t + 2y_2, s + 2x_2, -s - t + 2y_1, x_1, y_3, -s - t + 2y_4, -s - t + 2x_3 + 2x_4 \rangle, \quad (4.55)$$

$$I_8 = \langle s + t + 2y_2, s + 2x_2, -s - t + 2y_1, x_1, y_3, s + t + 2y_4, s + t - 2x_3 + 2x_4 \rangle, \quad (4.56)$$

under the parity symmetry (4.45), the primary ideals are permuted,

$$I_1 \leftrightarrow I_2, \quad I_3 \leftrightarrow I_4, \quad I_5 \leftrightarrow I_6, \quad I_7 \leftrightarrow I_8. \quad (4.57)$$

The irreducible numerator terms have the form,

$$x_3^m y_2^n x_4^a y_4^b. \quad (4.58)$$

And the integrand reduction method [25] determines that, the integrand basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where

$$\mathcal{B}_1 = \{x_3 y_2^5, y_2^6, x_3^4 y_2, x_3 y_2^4, y_2^5, x_3^4, x_3^3 y_2, x_3 y_2^3, y_2^4, x_3^3, x_3^2 y_2, x_3 y_2^2, y_2^3, x_3^2, x_3 y_2, y_2^2, x_3, y_2, 1\}, \quad (4.59)$$

and

$$\mathcal{B}_2 = \{x_4, x_3 x_4, x_3^2 x_4, x_3^3 x_4, x_4 y_2, y_4, x_3 y_4, x_3^2 y_4, x_3^3 y_4, x_3^4 y_4, y_2 y_4, x_3 y_2 y_4, y_2^2 y_4, x_3 y_2^2 y_4, y_2^3 y_4, x_3 y_2^3 y_4, y_2^4 y_4, x_3 y_2^4 y_4, y_2^5 y_4\}. \quad (4.60)$$

There are 19 terms in \mathcal{B}_1 .

Similarly, Define $\Omega = dD_1 \wedge \dots dD_7$. By solving congruence equations, we obtain rank-7 forms η_i , $i = 1, \dots, 8$ such that,

$$[\eta_i]_j = \delta_{ij} [\Omega]_j, \quad 1 \leq i, j \leq 8. \quad (4.61)$$

Again, to remove the spurious terms in \mathcal{B}_2 , we define,

$$v_1 = \eta_1 + \eta_3, \quad v_2 = \eta_2 + \eta_4, \quad v_3 = \eta_5 + \eta_6, \quad v_4 = \eta_7 + \eta_8. \quad (4.62)$$

We find that both v_1 and v_3 generate 4 IBPs, while v_2 and v_4 generate 3 IBPs. Again these IBPs are linearly independent, so our method generates 14 relations.

Furthermore, from the symmetry (4.46), we have,

$$2I_{\text{xbox}}[l_1 \cdot p_3] + I_{\text{xbox}}[l_2 \cdot p_2] = 0 + \dots, \quad (4.63)$$

$$2I_{\text{xbox}}[(l_1 \cdot p_3)(l_2 \cdot p_2)] + I_{\text{xbox}}[(l_2 \cdot p_2)^2] = 0 + \dots \quad (4.64)$$

These 2 relations are independent of the 14 IBP relations we obtained. Using these relations, we reduce the 19 terms in \mathcal{B}_1 to 3 terms,

$$I_{\text{xbox}}[1], \quad I_{\text{xbox}}[l_1 \cdot p_3], \quad I_{\text{xbox}}[(l_1 \cdot p_3)(l_2 \cdot p_2)]. \quad (4.65)$$

Again, there is one IBP relation missing in the pure $4D$ formalism. From FIRE [44], we have,

$$I_{\text{xbox}}[(l_1 \cdot p_3)(l_2 \cdot p_2)] = \frac{1}{16}(t+s)tI_{\text{xbox}}[1] - \frac{3}{8}(s+2t)I_{\text{xbox}}[l_1 \cdot p_3]. \quad (4.66)$$

Combine $14 + 2 + 1 = 17$ relations together, we reduce the integrand terms to two master integrals,

$$I_{\text{xbox}}[1], \quad I_{\text{xbox}}[l_1 \cdot p_3] \quad (4.67)$$

For example,

$$I_{\text{xbox}}[(l_2 \cdot p_2)^2] = -\frac{1}{8}t(s+t)I_{\text{xbox}}[1] + \frac{3}{4}(s+2t)I_{\text{xbox}}[l_1 \cdot p_3] + \dots, \quad (4.68)$$

$$\begin{aligned} I_{\text{xbox}}[(l_1 \cdot p_3)(l_2 \cdot p_2)^2] &= \frac{-t(s^2 + 3st + 2t^2)}{32}I_{\text{xbox}}[1] \\ &\quad + \frac{(3s^2 + 8st + 8t^2)}{16}I_{\text{xbox}}[l_1 \cdot p_3] + \dots, \end{aligned} \quad (4.69)$$

$$\begin{aligned} I_{\text{xbox}}[(l_2 \cdot p_2)^3] &= \frac{t(s^2 + 3st + 2t^2)}{16}I_{\text{xbox}}[1] \\ &\quad - \frac{(3s^2 + 8st + 8t^2)}{8}I_{\text{xbox}}[l_1 \cdot p_3] + \dots \end{aligned} \quad (4.70)$$

4.3 Slashed box

Our method also works for diagram with less than $DL - 1$ internal lines. In these cases, the coefficients α 's in (3.10) are not scalar functions, but differential forms. For example, consider the $4D$ slashed box with 4 massless legs, p_1, p_2, p_3 and p_4 . There are 5 denominators for slashed box integrals,

$$D_1 = l_1^2, \quad D_2 = (l_1 - p_2)^2, \quad D_3 = l_2^2, \quad D_4 = (l_2 - p_4)^2, \quad D_5 = (l_1 + l_2 + p_1)^2, \quad (4.71)$$

we use van Neerven-Vermaseren basis,

$$\begin{aligned} x_1 &= l_1 \cdot p_1, & x_2 &= l_1 \cdot p_2, & x_3 &= l_1 \cdot p_4, & x_4 &= l_1 \cdot \omega, \\ y_1 &= l_2 \cdot p_1, & y_2 &= l_2 \cdot p_2, & y_3 &= l_2 \cdot p_4, & y_4 &= l_2 \cdot \omega. \end{aligned} \quad (4.72)$$

where ω is the vector which is perpendicular to all external legs and $\omega^2 = tu/s$. The denominators have the parity symmetry,

$$x_4 \leftrightarrow -x_4, \quad y_4 \leftrightarrow -y_4. \quad (4.73)$$

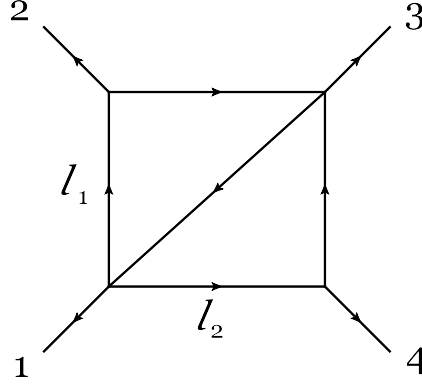


Figure 3. Planar slashed box with 4 massless legs

Define the ideal $I \equiv \langle D_1, \dots, D_7 \rangle$. The ISPs are $\{x_1, x_3, x_4, y_1, y_2, y_4\}$. Integrals with numerators linear in x_4 or y_4 are spurious.

The integrand basis for slashed box is $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ [25],

$$\begin{aligned} \mathcal{B}_1 = \{ & x_3^3 y_2, x_3^3 y_1, x_3^2 y_2^2, x_1 x_3^2 y_2, x_1 x_3^2 y_1, x_3^2 y_1^2, x_3 y_2^3, x_1 x_3 y_2^2, x_3 y_1 y_2^2, x_1^2 x_3 y_2, x_1 x_3 y_1 y_2, x_3 y_1^2 y_2, \\ & x_1^2 x_3 y_1, x_1 x_3 y_1^2, x_3 y_1^3, x_1 y_2^3, x_1^2 y_2^2, x_1 y_1 y_2^2, x_1^3 y_2, x_1^2 y_1 y_2, x_1 y_1^2 y_2, x_1^3 y_1, x_1^2 y_1^2, x_1 y_1^3, x_3^3, x_3^2 y_2, x_1 x_3^2, \\ & x_3^2 y_1, x_3 y_2^2, x_1 x_3 y_2, x_3 y_1 y_2, x_1^2 x_3, x_1 x_3 y_1, x_3 y_1^2, y_2^3, x_1 y_2^2, y_1 y_2^2, x_1^2 y_2, x_1 y_1 y_2, y_1^2 y_2, x_1^3, x_1^2 y_1, x_1 y_1^2, \\ & y_1^3, x_3^2, x_3 y_2, x_1 x_3, x_3 y_1, y_2^2, x_1 y_2, y_1 y_2, x_1^2, x_1 y_1, y_1^2, x_3, y_2, x_1, y_1, 1\}, \end{aligned} \quad (4.74)$$

and

$$\begin{aligned} \mathcal{B}_2 = \{ & x_4, x_1 x_4, x_1^2 x_4, x_3 x_4, x_1 x_3 x_4, x_3^2 x_4, x_4 y_1, x_1 x_4 y_1, x_1^2 x_4 y_1, x_3 x_4 y_1, x_1 x_3 x_4 y_1, x_3^2 x_4 y_1, \\ & x_4 y_1^2, x_1 x_4 y_1^2, x_4 y_1^3, x_4 y_2, x_1 x_4 y_2, x_1^2 x_4 y_2, x_4 y_1 y_2, x_1 x_4 y_1 y_2, x_4 y_1^2 y_2, y_4, x_1 y_4, x_1^2 y_4, x_1^3 y_4, x_3 y_4 \\ & , x_1 x_3 y_4, x_1^2 x_3 y_4, x_3^2 y_4, x_1 x_3^2 y_4, x_3^3 y_4, y_1 y_4, x_1 y_1 y_4, x_1^2 y_1 y_4, x_3 y_1 y_4, x_1 x_3 y_1 y_4, x_3^2 y_1 y_4, y_1^2 y_4, x_1 y_1^2 y_4, \\ & x_3 y_1^2 y_4, y_2 y_4, x_1 y_2 y_4, x_1^2 y_2 y_4, x_3 y_2 y_4, x_1 x_3 y_2 y_4, x_3^2 y_2 y_4, y_1 y_2 y_4, x_1 y_1 y_2 y_4, \\ & x_3 y_1 y_2 y_4, y_2^2 y_4, x_1 y_2^2 y_4, x_3 y_2^2 y_4\}. \end{aligned} \quad (4.75)$$

There are 59 terms in \mathcal{B}_1 and 52 terms in \mathcal{B}_2 . Terms in \mathcal{B}_2 are all spurious.

This diagram has the following symmetry,

$$l_1 \rightarrow -l_2 + p_4, \quad l_2 \rightarrow -l_1 + p_2, \quad (4.76)$$

$$p_1 \rightarrow p_3, \quad p_2 \rightarrow p_4, \quad p_3 \rightarrow p_1, \quad p_4 \rightarrow p_2. \quad (4.77)$$

The 4D crossed box cut has 4 branches,

$$I = I_1 \cap I_2 \cap I_3 \cap I_4, \quad (4.78)$$

where,

$$I_1 = \{x_2, y_3, x_1(-s-t) + y_1(-s-t) + 2x_3y_2, y_1(-\frac{t}{s}-1) - \frac{ty_2}{s} + y_4, \\ x_1(-\frac{t}{s}-1) - x_3 + x_4\}, \quad (4.79)$$

$$I_2 = \{x_2, y_3, x_1(-s-t) + y_1(-s-t) + 2x_3y_2, y_1(\frac{t}{s}+1) + \frac{ty_2}{s} + y_4, \\ x_1(\frac{t}{s}+1) + x_3 + x_4\}, \quad (4.80)$$

$$I_3 = \{x_2, y_3, x_1y_1(\frac{2t}{s}+2) + \frac{2tx_1y_2}{s} + tx_1 + ty_1 + 2x_3y_1, y_1(\frac{t}{s}+1) + \frac{ty_2}{s} + y_4, \\ x_1(-\frac{t}{s}-1) - x_3 + x_4\}, \quad (4.81)$$

$$I_4 = \{x_2, y_3, x_1y_1(\frac{2t}{s}+2) + \frac{2tx_1y_2}{s} + tx_1 + ty_1 + 2x_3y_1, y_1(-\frac{t}{s}-1) - \frac{ty_2}{s} + y_4, \\ x_1(\frac{t}{s}+1) + x_3 + x_4\} \quad (4.82)$$

Under the parity symmetry, the ideals are permuted as,

$$I_1 \leftrightarrow I_2, \quad I_3 \leftrightarrow I_4. \quad (4.83)$$

We have 5 denominators, so α_i 's in (3.10) are rank-2 differential forms. We use a basis for all possible rank-2 differential form,

$$\begin{aligned} \alpha^{(1)} &= dx_1 \wedge dx_3, & \alpha^{(2)} &= dx_1 \wedge dy_1, & \alpha^{(3)} &= dx_1 \wedge dy_2, & \alpha^{(4)} &= dx_3 \wedge dy_1, \\ \alpha^{(5)} &= dx_3 \wedge dy_2, & \alpha^{(6)} &= dy_1 \wedge dy_2, & \alpha^{(7)} &= dx_4 \wedge dy_4, & \alpha^{(8)} &= dx_1 \wedge dx_4 \\ \alpha^{(9)} &= dx_3 \wedge dx_4, & \alpha^{(10)} &= dy_1 \wedge dx_4, & \alpha^{(11)} &= dy_2 \wedge dx_4, & \alpha^{(12)} &= dx_1 \wedge dy_4 \\ \alpha^{(13)} &= dx_3 \wedge dy_4, & \alpha^{(14)} &= dy_1 \wedge dy_4, & \alpha^{(15)} &= dy_2 \wedge dy_4 \end{aligned} \quad (4.84)$$

Note that all components in $dD_1 \wedge \dots \wedge dD_5$ contains $dx_2 \wedge dy_3$. So we do not list rank-2 forms containing dx_2 or dy_3 . Now we define,

$$\Omega^{(i)} = \alpha^{(i)} \wedge dD_1 \wedge \dots \wedge dD_5, \quad 1 \leq i \leq 15 \quad (4.85)$$

Then we solve congruence equations to get 60 7-forms, $\omega_j^{(i)}$, $1 \leq i \leq 15$, $1 \leq j \leq 4$, such that,

$$[\omega_j^{(i)}]_k = \delta_{jk} [\Omega^{(i)}]_k. \quad (4.86)$$

We can use $\omega_j^{(i)}$'s to generate on-shell IBPs without doubled propagator. Again, to remove spurious terms, we define

$$\begin{aligned} v_{2i-1} &= \omega_1^{(i)} + \omega_2^{(i)} \\ v_{2i} &= \omega_3^{(i)} + \omega_4^{(i)}, \quad 1 \leq i \leq 15 \end{aligned} \quad (4.87)$$

Then all v_i 's are parity-even and we can use fv_i , $f \in \mathcal{B}_1$, to generate IBP relations.

However, the new feature for this diagram is that, we can use Remark. 1 to simplify the differential form and get more IBPs. For example,

$$v_{13} = -\frac{16(s(t(x_1 + y_1) + 2(x_1 + x_3)y_1) + 2tx_1(y_1 + y_2))}{s^2t^2(s + t)}\tilde{v}_{13}, \quad (4.88)$$

where,

$$\begin{aligned} \tilde{v}_{13} = & (s + t)(s + 2y_2)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 \\ & + (s + t)(t + 2x_3)dx_1 \wedge dx_2 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 \\ & + t(s + 2y_2)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4 \\ & - s(t + 2x_3)dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4. \end{aligned} \quad (4.89)$$

We can check that

$$[dD_i \wedge \tilde{v}_{13}] = 0, \quad 1 \leq i \leq 5 \quad (4.90)$$

So instead, we can use \tilde{v}_{13} to generate IBPs. In this manner, we get more IBPs. Similarly, v_{14} factorizes and we can define a new rank-7 form \tilde{v}_{14} for IBP generation. Other v_i 's do not have non-trivial factorization. Using all v_i (\tilde{v}_i)'s, we get 51 IBPs.

Furthermore, Ω^i themselves also have the factorization property. For example,

$$\Omega^{(1)} = -\frac{32x_4}{t^3(s + t)^3}\tilde{\omega}^{(1)}, \quad (4.91)$$

where,

$$\begin{aligned} \tilde{\Omega}^{(1)} = & -s(s + t)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 \\ & (s(y_4(t + x_1 + x_3) - x_4(y_1 + y_3)) + t(y_4(x_1 + x_2) - x_4(y_1 + y_2))) \\ & + stdx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4 \\ & (s(y_4(x_3 - x_1) + x_4(y_1 - y_3)) + t(x_4(y_1 + y_2) - y_4(x_1 + x_2))) \\ & - t(s + t)(s(t(y_1 - y_3) - 2x_1y_3 + 2x_3y_1) + t(t(y_1 + y_2) + 2(x_3(y_1 + y_2) - y_3(x_1 + x_2)))) \\ & dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3. \end{aligned} \quad (4.92)$$

We can verify that,

$$[dD_i \wedge \tilde{\Omega}^{(1)}] = 0, \quad 1 \leq i \leq 5 \quad (4.93)$$

So we can use $\tilde{\Omega}^{(1)}$ to generate IBPs. Similarly, $\Omega^{(6)}$, $\Omega^{(8)}$, $\Omega^{(9)}$, $\Omega^{(14)}$ and $\Omega^{(15)}$ also factorize. Using $\tilde{\Omega}$ forms, we get 4 more independent IBPs.

Note that although $\Omega^{(1)}$ itself has the form $\alpha \wedge dD_1 \wedge \dots \wedge dD_5$, where α is a polynomial-valued differential form. However, $\tilde{\Omega}^{(1)}$ cannot be expressed as a product of polynomial-valued form and $dD_1 \wedge \dots \wedge dD_5$. So $\tilde{\Omega}^{(1)}$ does not satisfy the conditions in Theorem. 1 and there is no way to solve the congruence equation,

$$[\tilde{\Omega}_j^{(1)}]_k = \delta_{jk}[\tilde{\Omega}^{(1)}]_k, \quad 1 \leq k \leq 4 \quad (4.94)$$

to get more differential forms.

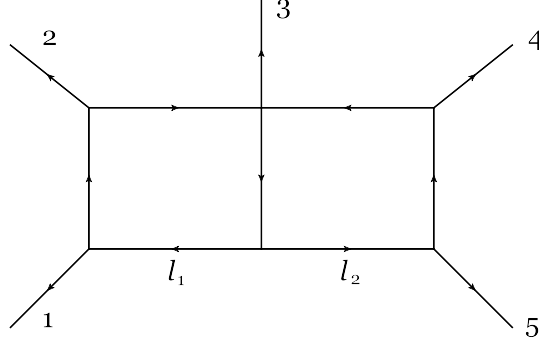


Figure 4. Planar double box with 5 massless legs

In summary, from differential forms, we get $51 + 4 = 55$ IBP relations. Furthermore, using the symmetry condition (4.76), we have,

$$I_{\text{slashed}}[l_2 \cdot p_1] = -I_{\text{slashed}}[l_1 \cdot p_3] + \frac{t}{2} I_{\text{slashed}}[1]. \quad (4.95)$$

So we have $59 - 55 - 1 = 3$ integrals left,

$$I_{\text{slashed}}[1], \quad I_{\text{slashed}}[l_1 \cdot p_1], \quad I_{\text{slashed}}[(l_1 \cdot p_1)^2] \quad (4.96)$$

From FIRE [44], there are two missing IBPs,

$$I_{\text{slashed}}[l_1 \cdot p_1] = -\frac{st}{2u} I_{\text{slashed}}[1], \quad (4.97)$$

$$I_{\text{slashed}}[(l_1 \cdot p_1)^2] = \frac{s^2 t^2}{4u^2} I_{\text{slashed}}[1]. \quad (4.98)$$

So the 59 integrand terms reduce to 1 master integral, $I_{\text{slashed}}[1]$. For example,

$$I_{\text{slashed}}[l_1 \cdot p_4] = -\frac{t}{2} I_{\text{slashed}}[1], \quad (4.99)$$

$$I_{\text{slashed}}[(l_1 \cdot p_1)(l_1 \cdot p_4)] = \frac{st^2}{4u} I_{\text{slashed}}[1], \quad (4.100)$$

$$I_{\text{slashed}}[(l_1 \cdot p_1)(l_2 \cdot p_1)] = \frac{s^2 t^2}{2u^2} I_{\text{slashed}}[1], \quad (4.101)$$

$$I_{\text{slashed}}[(l_2 \cdot p_1)(l_2 \cdot p_2)] = \frac{s^2 t}{4u} I_{\text{slashed}}[1]. \quad (4.102)$$

4.4 Turtle box

Now consider the $4D$ two-loop turtle box with 5 massless legs, p_1, p_2, p_3, p_4 and p_5 . This system is considerably more difficult than the 4-point two-loop cases, since the kinematics is complicated. The two loop momenta are l_1 and l_2 . There are 7 denominators for crossed box integrals,

$$\begin{aligned} D_1 &= l_1^2, & D_2 &= (l_1 - p_1)^2, & D_3 &= (l_1 - p_1 - p_2)^2, \\ D_4 &= (l_2 - p_5)^2, & D_5 &= (l_2 - p_4 - p_5)^2, & D_6 &= l_2^2, & D_7 &= (l_1 + l_2)^2. \end{aligned} \quad (4.103)$$

In this case, we find that it is easier to calculate differential forms and IBP identity in spinor helicity formalism, and then convert the result to van Neerven-Vermaseren basis in the final step. Define,

$$l_1^\mu = \alpha_1 p_1^\mu + \alpha_2 p_2^\mu + \frac{s_{12}\alpha_3}{\langle 14 \rangle [42]} \frac{[1|\gamma^\mu|2\rangle}{2} + \frac{s_{12}\alpha_4}{\langle 24 \rangle [41]} \frac{[2|\gamma^\mu|1\rangle}{2}, \quad (4.104)$$

$$l_2^\mu = \beta_1 p_4^\mu + \beta_2 p_5^\mu + \frac{s_{12}\beta_3}{\langle 41 \rangle [15]} \frac{[4|\gamma^\mu|5\rangle}{2} + \frac{s_{12}\beta_4}{\langle 51 \rangle [14]} \frac{[5|\gamma^\mu|4\rangle}{2}. \quad (4.105)$$

Furthermore, to simplify the computation, we use *momentum-twistor* variables [47, 48] for s_{ij} , $\langle i, j \rangle$ and $[i, j]$. The advantage is that all constraints like momentum conservation and Schouten identities are resolved in *momentum-twistor* variables.

The ISPs are

$$a = l_1 \cdot p_4, \quad b = l_1 \cdot p_5, \quad c = l_2 \cdot p_1, \quad d = l_2 \cdot p_2, \quad (4.106)$$

The integrand basis contains 32 terms,

$$\mathcal{B} = \{b^4 c, b^4 d, bcd^3, bd^4, ab^3, b^4, b^3 c, b^3 d, bcd^2, bd^3, cd^3, d^4, ab^2, b^3, b^2 c, b^2 d, bcd, bd^2, cd^2, d^3, ab, ad, b^2, bc, bd, cd, d^2, a, b, c, d, 1\}, \quad (4.107)$$

Note that for 5-point kinematics, there exists no vector ω perpendicular to all external legs. So it is not obvious to find spurious terms directly from the integrand basis. However, we have the following identities,

$$\int \frac{d^4 l_1}{(2\pi)^2} \frac{d^4 l_2}{(2\pi)^2} \frac{\epsilon(l_1, l_2, p_1, p_2) g(l_2)}{D_1 \dots D_7} = 0, \quad (4.108)$$

$$\int \frac{d^4 l_1}{(2\pi)^2} \frac{d^4 l_2}{(2\pi)^2} \frac{\epsilon(l_2, l_1, p_4, p_5) f(l_1)}{D_1 \dots D_7} = 0, \quad (4.109)$$

because of the parity properties for the sub-diagrams. Here $f(l_1)$ and $g(l_2)$ are arbitrary Lorentz-invariant functions of l_1 and l_2 , respectively.

There are 6 branches for cut solutions,

$$I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6. \quad (4.110)$$

Similarly, Define $\omega = dD_1 \wedge \dots \wedge dD_7$. By solving congruence equations, we obtain rank-7 forms η_i , $i = 1, \dots, 6$ such that,

$$[\eta_i]_j = \delta_{ij} [\Omega]_j, \quad 1 \leq i, j \leq 6. \quad (4.111)$$

We find that each of the first 4 differential forms η_1, \dots, η_4 generates 3 IBPs, while each of the differential forms η_5 and η_6 generate 4 IBPs. These relations are linearly independent, so there are 24 IBPs in total. Furthermore, the identities (4.108) provides two more independent identities. So we have $32 - 26 = 6$ integrals left,

$$\begin{aligned} &I_{\text{turtle}}[1], \quad I_{\text{turtle}}[l_1 \cdot p_4], \quad I_{\text{turtle}}[l_1 \cdot p_5], \quad I_{\text{turtle}}[l_2 \cdot p_1], \\ &I_{\text{turtle}}[l_2 \cdot p_2], \quad I_{\text{turtle}}[(l_1 \cdot p_4)(l_2 \cdot p_2)] \end{aligned} \quad (4.112)$$

There is a subtlety for the master integrals of turtle diagram. For the D -dimensional cases, there are 3 master integrals, $I_{\text{turtle}}[1]$, $I_{\text{turtle}}[l_1 \cdot p_4]$ and $I_{\text{turtle}}[l_1 \cdot p_5]$. However, for $D = 4$, there are only 2 master integral $I_{\text{turtle}}[1]$, $I_{\text{turtle}}[l_1 \cdot p_4]$, because of an integrand reduction relation in $4D$. Since we start with the $4D$ minimal integrand, this additional relation is already incorporated. Then using 4 additional IBPs from FIRE [44],

$$\begin{aligned} I_{\text{turtle}}[l_2 \cdot p_1] &= I_{\text{turtle}}[l_1 \cdot p_5], \\ I_{\text{turtle}}[l_2 \cdot p_2] &= \frac{s_{25}}{s_{14}} I_{\text{turtle}}[l_1 \cdot p_4], \\ I_{\text{turtle}}[(l_1 \cdot p_4)(l_2 \cdot p_2)] &= \frac{s_{12}s_{45}}{8} I_{\text{turtle}}[1] + \frac{s_{25}}{4} I_{\text{turtle}}[l_1 \cdot p_4] - \frac{s_{24}}{4} I_{\text{turtle}}[l_1 \cdot p_5], \\ I_{\text{turtle}}[(l_1 \cdot p_5)(l_2 \cdot p_2)] &= \frac{s_{15}s_{25}}{4s_{14}} I_{\text{turtle}}[l_1 \cdot p_4] - \frac{s_{25}}{4} I_{\text{turtle}}[l_1 \cdot p_5]. \end{aligned} \quad (4.113)$$

Including these missing IBP relations, we reduce all integrand terms to the master integrals $I_{\text{turtle}}[1]$, $I_{\text{turtle}}[l_1 \cdot p_4]$. For example,

$$\begin{aligned} I_{\text{turtle}}[l_1 \cdot p_5] &= -\frac{4s_{15}(s_{12} + s_{15} - s_{34})}{F} I_{\text{turtle}}[(l_1 \cdot p_4)] \\ &\quad - \frac{s_{15}(s_{23}s_{34} + (s_{15} - s_{34})s_{45} + s_{12}(s_{15} - s_{23} + 2s_{45}))}{F} I_{\text{turtle}}[1] + \dots, \end{aligned} \quad (4.114)$$

$$\begin{aligned} I_{\text{turtle}}[(l_1 \cdot p_4)(l_2 \cdot p_1)] &= -\frac{1}{2F} s_{15}(s_{23}s_{34} + (s_{15} - s_{34})s_{45} + s_{12}(s_{15} - s_{23} + 2s_{45})) I_{\text{turtle}}[(l_1 \cdot p_4)] \\ &\quad - \frac{1}{4F} s_{15}(s_{15} - s_{23} + s_{45})(s_{23}s_{34} + (s_{15} - s_{34})s_{45} + s_{12}(s_{15} - s_{23} + 2s_{45})) I_{\text{turtle}}[1] \\ &\quad + \dots, \end{aligned} \quad (4.115)$$

$$\begin{aligned} I_{\text{turtle}}[(l_1 \cdot p_4)^2(l_1 \cdot p_5)] &= -s_{15}(s_{12} + s_{15} - s_{34})(s_{15} - s_{23} + s_{45})^2 I_{\text{turtle}}[l_1 \cdot p_4] \\ &\quad - \frac{1}{4} s_{15}(s_{15} - s_{23} + s_{45})^2 (s_{23}s_{34} + (s_{15} - s_{34})s_{45} + s_{12}(s_{15} - s_{23} + 2s_{45})) I_{\text{turtle}}[1] + \dots, \end{aligned} \quad (4.116)$$

$$I_{\text{turtle}}[(l_1 \cdot p_4)(l_2 \cdot p_1)(l_2 \cdot p_2)] = 0 + \dots, \quad (4.117)$$

where the polynomial F is,

$$F = 2(2s_{15}^2 + (-2s_{23} - 2s_{34} + s_{45})s_{15} + s_{12}(s_{15} - s_{23}) + s_{34}(s_{23} - s_{45})) \quad (4.118)$$

The complete result for $4D$ on-shell turtle box IBPs can be downloaded at <http://www.nbi.dk/~zhang/>

It is interesting to compare our result to the result from GKK method [51]. GKK method determines that in $D = 4 - 2\epsilon$ dimension, there are 15 IBP generating vectors $v_{\text{GKK}}^{(i)}$, $i = 1, \dots, 15$, without doubled propagator. However, in the $4D$ on-shell limit, we explicitly verified that on each of the 6 branches, for all 15 vectors the dual form $\omega_{\text{GKK}}^{(i)}$ is proportional to Ω . Hence, in the $4D$ on-shell limit, the 15 vectors are generated by our six local forms η_j , $j = 1, \dots, 6$.

5 Conclusion

In this paper, we invent a new method to generate integration-by-part identities from the viewpoint of differential geometry. The generating vector for IBP identities are reformulated as differential forms, via Poincaré dual. Then by techniques of differential geometry,

the geometric meaning of generating vectors for IBPs without doubled propagator is clear: *they are dual to the normal direction of the unitarity-cut solution.*

By using the wedge product and congruence equations over cut branches, suitable differential forms to generate IBP without doubled propagator are obtained. Our algorithm is realized by our computational algebraic geometry package, MATHEMATICAM2.

We tested our algorithm on several $4D$ two-loop examples. The algorithm is very efficient in generating the analytic on-shell part of IBP identities. For example, our program obtains the analytic on-shell IBPs of 5-point turtle diagram, in about one hour on our laptop.

Following our discoveries, there are several interesting future directions,

- The extension of our formalism to $D = (4 - 2\epsilon)$ -dimension. Apparently, the differential forms are not directly defined in non-integer dimensions. But we expect that this difficulty can be circumvented by considering our formalism in various integer-valued dimensions, and then combine the results by an analytic continuation. In general, the D -dimensional unitarity cut solution has a simpler structure than its $4D$ counterpart, so we expect that the discussion on the local properties of differential forms can be simplified in D -dimensional cases.
- The beyond-on-shell part of IBP. For the purpose of finding the contour weights in maximal unitarity [14], the algorithm is enough since it aims at the on-shell part. It is interesting to see that how to go steps further by releasing the cut constraints recursively.
- Combination of our differential form method with the classic IBP generating algorithm like Laporta. Our method focuses on the IBP relations without doubled propagator, while other algorithms can recover all the IBP relations. Even before applying the sophisticated congruence method, it is straightforward to calculate the differential form $\Omega = dD_1 \wedge \dots \wedge dD_k$ analytically, and this form itself generate a lot of IBPs without doubled propagator. We expect that the ingredients of our method can be incorporated current IBP generating programs to speed up the computation.

Acknowledgement

We thank Simon Badger, Emil J. Bjerrum-Bohr, Spencer Bloch, Simon Caron-Huot, Poul Damgaard, Hjalte Frellesvig, Rijun Huang, David Kosower, Kasper Larsen and Mads Sogaard for useful discussion on this project. We express special gratitude to Simon Caron-Huot for his participation in the early stage of this paper and careful reading of this paper in the draft stage. We also thank David Kosower and IPhT, Saclay for the hospitality during YZ's visit. YZ is supported by Danish Council for Independent Research-Natural Science (FNU) grant 11-107241.

A Review of mathematical notations

The denominators D_1, \dots, D_k for a Feynman integral, generates an ideal in the polynomial ring $R = \mathbb{C}[x_1, \dots, x_{DL}]$,

$$I = \langle D_1, \dots, D_k \rangle. \quad (\text{A.1})$$

The *cut solution* is the zero locus of all denominators,

$$\mathcal{S} = \mathcal{Z}(I) = \{(a_1, \dots, a_{DL}) \in \mathbb{C}^{LD} \mid D_1(a_1, \dots, a_{DL}) = \dots = D_k(a_1, \dots, a_{DL}) = 0\}. \quad (\text{A.2})$$

In many cases, the cut solution contains several branches, in mathematical language, the ideal I has a primary decomposition,

$$I = I_1 \cap \dots \cap I_n, \quad (\text{A.3})$$

So correspondingly, the cut solution decomposes into several irreducible branches,

$$\mathcal{S} = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_n, \quad (\text{A.4})$$

where $\mathcal{S}_j = \mathcal{Z}(I_j)$.

By Hilbert's Nullstellensatz, if a polynomial f vanishes everywhere on $\mathcal{Z}(I)$, then $f \in \sqrt{I}$. Here \sqrt{I} is the radical of I ,

$$\sqrt{I} = \{f \mid f^s \in I, s \in \mathbb{N}\}. \quad (\text{A.5})$$

\sqrt{I} is also an ideal and $I \subset \sqrt{I}$. If $I = \sqrt{I}$, we call I a radical ideal.

The integrand N can be reduced by polynomial division towards the denominators, via Gröbner basis

$$N = \Delta + \sum_i^k f_i D_i, \quad (\text{A.6})$$

where the remainder Δ , is the *integrand basis*. We call monomials in Δ *irreducible numerators*.

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